SI 2013 Mark Scheme Q1

(i)	$y=\sqrt{x} \;\Rightarrow\; y^2+3y-\frac{1}{2}=0$	**B1**	for a correct quadratic eqn. in y or $\sqrt{x}$
	$(2y+3)^2=11$	**M1**	for use of a method for solving a quadratic eqn. (compl^{g}. the square, formula, etc.) If candidate fails to obtain a numerical answer for y (correct or not) then **M0**
	$y=\dfrac{-3\pm\sqrt{11}}{2}$	**A1**	
	$y\geq 0 \;\Rightarrow\; \sqrt{x}=\dfrac{\sqrt{11}-3}{2}$	**M1**	for clearly choosing the correct root: **FT** *provided* they have 1 $+_{ve}$ and 1 $-_{ve}$ root to choose from
	$x=\left(\dfrac{\sqrt{11}-3}{2}\right)^2$ or $\dfrac{20-6\sqrt{11}}{4}$ or $5-\frac{3}{2}\sqrt{11}$	**A1**	**5**
(ii) (a)	$y=\sqrt{x+2}$	**M1**	for clear indication of this substitution (or equivalent)
	$y^2+10y-24=0$	**A1**	for a correct quadratic
	y or $\sqrt{x+2}$ $=-12,\ 2$	**M1**	for solution method of a suitable quadratic
	$y\geq 0 \;\Rightarrow\; \sqrt{x+2}=2$	**M1**	for choosing valid root: **FT** *provided* they have 1 $+_{ve}$ and 1 $-_{ve}$ root to choose from
	$x=2$	**A1**	**5**
(ii) (b)	$y=\sqrt{2x^2-8x-3}$	**M1**	for clear indication of this substitution (or equivalent)
	$y^2+2y-15=0$	**A1**	for a correct quadratic
	y or $\sqrt{2x^2-8x-3}$ $=-5,\ 3$	**M1**	for solution of a suitable quadratic
	$y\geq 0 \;\Rightarrow\; \sqrt{2x^2-8x-3}=3$	**M1**	for choosing valid root: **FT** *provided* they have 1 $+_{ve}$ and 1 $-_{ve}$ root to choose from
	$2x^2-8x-3=9 \;\Rightarrow\; x^2-4x-6=0$	**M1A1**	for obtaining and solving a quadratic eqn. in x; A1 for the correct quadratic
	$x=2\pm\sqrt{10}$	**A1**	**7**

$x = 2 \pm \sqrt{10} \;\Rightarrow\; x^2 = 14 \pm 4\sqrt{10}$ **M1** for checking attempt (for at least one of the answers found)

so $x^2 - 4x - 9 = -3$ & $2x^2 - 8x - 3 = 9$

$\Rightarrow$ (both cases) $-3 + \sqrt{9} = 0$ **A1A1** one for each clearly shown (with working)

ALTERNATIVELY For validity, $2x^2 - 8x - 3 \geq 0$ also **M1** i.e. $(x-2)^2 \geq \frac{11}{2}$ **A1**

Since $(x-2)^2 = 10 > \frac{11}{2}$ both solns. valid **E1**

$\boxed{3}$

ALTERNATIVES

(i) $3\sqrt{x} = \frac{1}{2} - x$ and squaring **M1** $x^2 - 10x + \frac{1}{4} = 0$ **A1** correct quadratic **M1** for solution of a suitable quadratic **A1** $x = 5 \pm \frac{3}{2}\sqrt{11}$

However, *both* these roots are positive, so the final mark will be **E1** for checking both, with working, and correctly discarding the unsuitable answer

(ii) (a) $10\sqrt{x+2} = 22 - x$ and squaring **M1** $x^2 - 144x + 284 = 0$ **A1** correct quadratic **M1** for solution of a suitable quadratic **A1** $x = 142, 2$

E1 for checking both, with working, and correctly discarding the unsuitable answer (e.g. $x = 142$ gives LHS > 0 but RHS < 0 would suffice)

(ii) (b) $\sqrt{2x^2 - 8x - 3} = 9 + 4x - x^2$ and squaring **M1** $x^4 - 8x^3 - 4x^2 + 80x + 84 = 0$ **A1** correct quartic

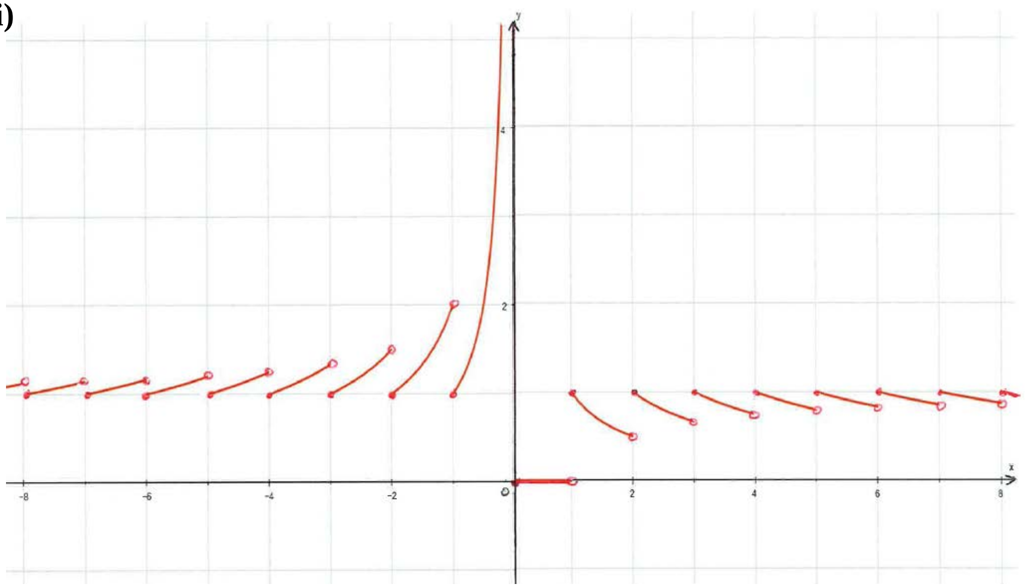
$(x-2)^4 - 28(x-2)^2 + 180 = 0$ **M1A1** $\Rightarrow$ $(x-2)^2 = 10, 18$ **M1A1**

Now $\sqrt{2}\sqrt{(x-2)^2 - \frac{11}{2}} = 13 - (x-2)^2 \;\Rightarrow\; \frac{11}{2} \leq (x-2)^2 \leq 13$ **M1A1A1** so the only valid solutions arise from $(x-2)^2 = 10$ and $x = 2 \pm \sqrt{10}$ **A1**

However, I cannot see candidates making this approach work. **M1A1** for getting the correct quartic may be all they can reasonably get.
Attempts to find linear factors (by the *Factor Theorem*, for instance) will go nowhere.
Some may attempt to find a pair of quadratic factors: $(x^2 + Ax + B)(x^2 + Cx + D) \equiv x^4 + (A+C)x^3 + (AC+B+D)x^2 + (AD+BC)x + BD = 0$ and compare terms ($A + C = -8$, $AC + B + D = -4$, $AD + BC = 80$ and $BD = 84$) but I do not want them to have any marks unless they can get to (by guessing/verifying … divine inspiration?) $(x^2 - 4x - 6)(x^2 - 4x - 14)$, at which point I would award them the next **M1A1** & **M1A1**.
They now have four answers to check for and I would propose a **B1** for each correctly checked (with working) and accepted/rejected appropriately.

SI 2013 Mark Scheme Q2

(i)



G1 Lots of "unit" segments

G1 $\left.\begin{array}{l}\text{LH}\\\text{RH}\end{array}\right\}$ end clearly $\left\{\begin{array}{l}\text{included}\\\text{excluded}\end{array}\right.$

G1 Each segment looks like a portion of a reciprocal curve

G1 Essentially correct $0 \le x \le 3$

G1 Essentially correct $-3 \le x < 0$

Ignore endpoint issues for these last two Gs

5

(ii) Note that, for $n \le x < n+1$, $\lfloor x \rfloor = n$ so f(x) = $\frac{n}{x}$. Also, $\frac{n}{n+1} < \mathrm{f}(x) \le 1$ for $x > 0$, and f(x) ≥ 1 for $x < 0$, so …

f(x) = $\frac{7}{12}$ only in [1, 2). **E1** Sketch may show it so

$\mathrm{f}(x) = \frac{1}{x} = \frac{7}{12} \Rightarrow x = \frac{12}{7}$ **B1** B0 if extra answers appear

2

Similarly, $\frac{n}{n+1} > \frac{17}{24} \Rightarrow 24n > 17n + 17 \Rightarrow n > 2\frac{3}{7}$, i.e. $n \ge 3$; so f(x) = $\frac{17}{24}$ only in [1, 2) and [2, 3).

In [1, 2), $\mathrm{f}(x) = \frac{1}{x} = \frac{17}{24} \Rightarrow x = \frac{24}{17}$ **B1**

In [2, 3), $\mathrm{f}(x) = \frac{2}{x} = \frac{17}{24} \Rightarrow x = \frac{48}{17}$ **B1** Give max. B1 if extra answers appear

2

Now, for $x < 0$, $1 \le f(x) < \dfrac{n}{n+1}$, and $\dfrac{-n}{-n-1} < \dfrac{4}{3}$ $\Rightarrow$ $-4n - 4 > -3n$ $\Rightarrow$ $n < -4$; so $f(x) = \frac{4}{3}$ only in $[-4, -3)$, $[-3, -2)$, $[-2, -1)$ and $[-1, 0)$.

$f(-3) = 1$ so no solution in $[-4, -3)$	**B1**	Possibly implicitly, if just not there
In $[-3, -2)$, $f(x) = \dfrac{-3}{x} = \dfrac{4}{3}$ $\Rightarrow$ $x = -\frac{9}{4}$	**B1**	
In $[-2, -1)$, $f(x) = \dfrac{-2}{x} = \dfrac{4}{3}$ $\Rightarrow$ $x = -\frac{3}{2}$	**B1**	
In $[-1, 0)$, $f(x) = \dfrac{-1}{x} = \dfrac{4}{3}$ $\Rightarrow$ $x = -\frac{3}{4}$	**B1**	Give max. B1B1 if extra answers appear

4

(iii) $\dfrac{n}{n+1} > \dfrac{9}{10}$ for $n > 9$ so …

$f(x_{max}) = \frac{9}{10}$ in $[8, 9)$	**E1**	
and $f(x) = \dfrac{8}{x} = \dfrac{9}{10}$ $\Rightarrow$ $x = \frac{80}{9}$	**B1**	NB $f(10) = 1$, so $x \ne 10$

2

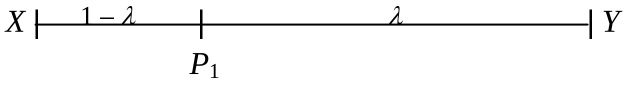
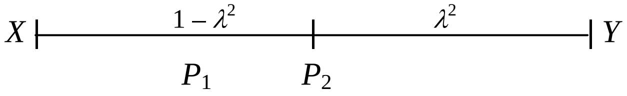
$f(x) = c$ has exactly n roots for …

for $x > 0$: $\dfrac{n}{n+1} < c \le \dfrac{n+1}{n+2}$	**B1B1**	LHS; RHS
for $x < 0$: $\dfrac{n+1}{n} \le c < \dfrac{n}{n-1}$, $n \ge 2$	**B1B1**	LHS; RHS
$c \ge 2$, $n = 1$	**B1**	

5

SI 2013 Mark Scheme Q3

(i)	$X * Y = Y * X \iff \lambda\mathbf{x} + (1-\lambda)\mathbf{y} = \lambda\mathbf{y} + (1-\lambda)\mathbf{x}$	**M1**	Including correct $Y * X = \lambda\mathbf{y} + (1-\lambda)\mathbf{x}$	**3**
	$\iff (2\lambda - 1)(\mathbf{x} - \mathbf{y}) = \mathbf{0}$	**M1**		
	(Since $\mathbf{x} \neq \mathbf{y}$) $\lambda = \frac{1}{2}$	**A1**		
(ii)	$(X * Y) * Z = \lambda(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) + (1-\lambda)\mathbf{z}$	**M1**		
	$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$	**A1**		
	and			
	$X * (Y * Z) = \lambda\mathbf{x} + (1-\lambda)[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}]$	**M1**		
	$= \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$	**A1**		
	$(X * Y) * Z - X * (Y * Z) = \lambda(1-\lambda)(\mathbf{x} - \mathbf{z})$	**M1**		**6**
	The two are distinct provided $\lambda \neq 0, 1$ or $X \neq Z$	**A1**		
(ii)	$(X * Y) * Z = \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$			
	$(X * Z) * (Y * Z) = [\lambda\mathbf{x} + (1-\lambda)\mathbf{z}] * [\lambda\mathbf{y} + (1-\lambda)\mathbf{z}]$	**M1**		
	$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{z} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$			
	$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$		**A1** (and the two are always equal)	
	$X * (Y * Z) = \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$			
	$(X * Y) * (X * Z) = [\lambda\mathbf{x} + (1-\lambda)\mathbf{y}] * [\lambda\mathbf{x} + (1-\lambda)\mathbf{z}]$	**M1**		
	$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + \lambda(1-\lambda)\mathbf{x} + (1-\lambda)^2\mathbf{z}$			
	$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$		**A1**	**5**
	Hence $X * (Y * Z) = (X * Y) * (X * Z)$	**E1**	Conclusion must be stated (before or after proof)	

For $0 < \lambda < 1$, P_1 cuts XY in the ratio $(1 - \lambda) : \lambda$	**B1**	
		
	**M1**	Iterating
We see that P_n cuts XY in the ratio $1 - \lambda^n : \lambda^n$	**A1A1**	Any correct form
$P_{n+1} = P_n * Y = \lambda\, \mathbf{p}_{\boldsymbol{n}} + (1 - \lambda)\mathbf{y}$ $= \lambda \left\{ \left(\lambda^n\right)\mathbf{x} + \left(1 - \lambda^n\right)\mathbf{y} \right\} + \left(1 - \lambda\right)\mathbf{y}$ $= \lambda^{n+1}\mathbf{x} + \left(\lambda - \lambda^{n+1} + 1 - \lambda\right)\mathbf{y}$ $= \lambda^{n+1}\mathbf{x} + \left(1 - \lambda^{n+1}\right)\mathbf{y}$		
and proof follows by induction	**M1A1**	for attempt at an inductive proof; fully correct

6

SI 2013 Mark Scheme Q4

(i)	$\int \tan^n x.\sec^2 x \, dx = \left[\frac{1}{n+1}\tan^{n+1} x\right]$	**M1**	May be done via a substn. such as $t = \tan x$ (or by "parts")	**2**
	$= \frac{1}{n+1}$	**A1**	**ANSWER GIVEN**	
	$\int \sec^n x.\tan x \, dx = \int \sec^{n-1} x.\sec x \tan x \, dx$	**M1**	May be done via a substn. such as $s = \sec x$ (or by parts)	
	$= \left[\frac{1}{n}\sec^n x\right]$	**A1**		
	$= \frac{\left(\sqrt{2}\right)^n - 1}{n}$	**A1**	**ANSWER GIVEN**	**3**
(ii) (a)	$\int_0^{\pi/4} x\sec^4 x \tan x \, dx = \left[x.\frac{\sec^4 x}{4}\right]_0^{\pi/4} - \int_0^{\pi/4} \frac{\sec^4 x}{4} dx$	**M1A1A1**	for appropriate use of *parts*; correct (1st, 2nd)	
	$= \frac{\pi}{4} - \frac{1}{4}J$ where $J = \int_0^{\pi/4} \sec^4 x \, dx$			
	$J = \int_0^{\pi/4} \sec^2 x \, dx + \int_0^{\pi/4} \sec^2 x \tan^2 x \, dx$	**M1**	for use of $\sec^2 x = 1 + \tan^2 x$ to split up the integral	
	$= \left[\tan x + \tfrac{1}{3}\tan^3 x\right]$	**A1 A1**		
	$= \frac{4}{3}$		NB Limits are ignored until the end, when numerical answers need to appear	
	Thus $I = \frac{\pi}{4} - \frac{1}{3}$	**A1**		**7**

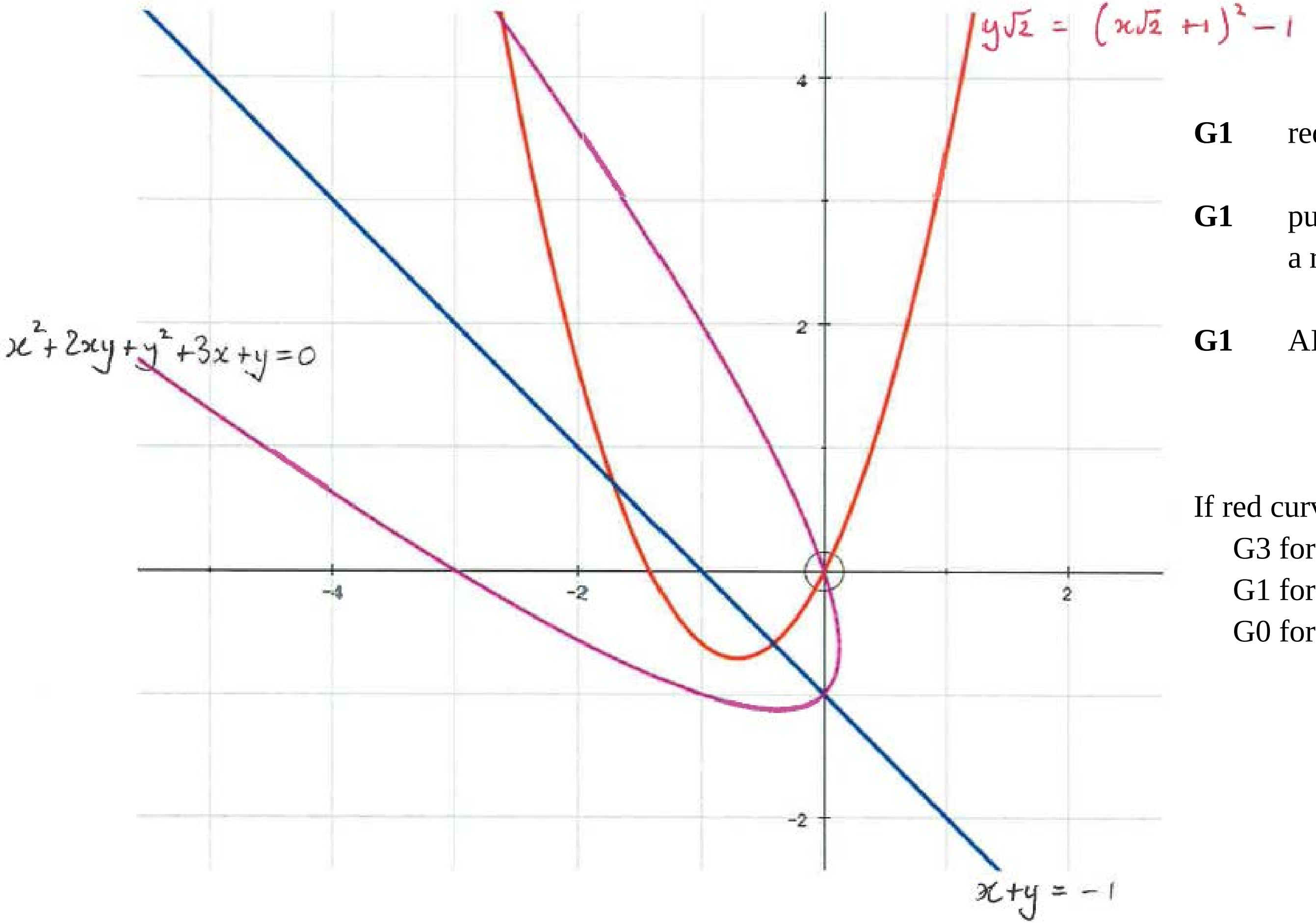
(ii) (b) $\int x^2(\sec^2 x.\tan x)\,dx$	**M1**	for appropriate splitting and attempted use of *integration by parts*
$= \left[x^2.\tfrac{1}{2}\tan^2 x\right] - \int 2x.\tfrac{1}{2}\tan^2 x\ dx$	**A1A1**	
$= \dfrac{\pi^2}{32} - \int x(\sec^2 x - 1)$	**M1**	for use of $\tan^2 x = \sec^2 x - 1$
$= \dfrac{\pi^2}{32} - K + \int x\,dx$ where $K = \int_0^{\pi/4} x\sec^2 x\ dx$		
$K = [x.\tan x] - \int \tan x\ dx$	**M1**	
$= x\tan x - \ln(\sec x)$	**B1**	for $\int \tan x\,dx = \ln(\sec x)$
$= \dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$	**A1**	
Thus $I = \dfrac{\pi^2}{32} - \left(\dfrac{\pi}{4} - \dfrac{1}{2}\ln 2\right) + \dfrac{\pi^2}{32}$		
$= \dfrac{\pi^2}{16} - \dfrac{\pi}{4} + \dfrac{1}{2}\ln 2$	**A1**	
Note that there are many ways to split these integrals in (ii) into parts.		**8**

SI 2013 Mark Scheme Q5

(i)	For $k = 0$: $x^2 + 3x + y^2 + y = 0$			
	$\left(x + \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \left(\frac{1}{2}\sqrt{10}\right)^2$	**M1**	for completing the square for both x and y	**4**
	giving a CIRCLE	**G1**	for a circle drawn	
	thro' (0, 0), (0, –1) & (–3, 0)	**B1B1**	passing thro' the origin; other 2 intercepts correct (noted on sketch or separately)	
(ii)	For $k = \frac{10}{3}$: $(3x + y)(x + 3y + 3) = 0$	**B1**	Must be the full thing (no marks for just factorising the given quadratic)	
	giving a LINE-PAIR	**G1**	for two (intersecting) lines drawn	
	Lines $y = -3x$ & $x + 3y = -3$		Statement of eqns. not actually required	
	1st line thro' O with $-_{ve}$ gradient	**B1**		**5**
	2nd line not thro' O with $-_{ve}$ gradient	**M1**		
	thro' (0, –1) & (–3, 0)	**A1**	for both stated or noted on sketch	
(iii)	For $k = 2$: $(x + y)^2 + 3x + y = 0$			
	When $\theta = 45^{\text{o}}$,			
	$x + y = X\sqrt{2}$ and $y - x = Y\sqrt{2}$	**B1**	noted or used anywhere	
	$\Rightarrow x = \dfrac{X - Y}{\sqrt{2}}$ and $y = \dfrac{X + Y}{\sqrt{2}}$	**M1A1**		
	$(x + y)^2 + 3x + y = 0$ becomes …			
	$2X^2 + \dfrac{3X - 3Y}{\sqrt{2}} + \dfrac{X + Y}{\sqrt{2}} = 0$	**M1**	for eliminating both x and y for X and Y	
	$\Rightarrow 2X^2 + 2\sqrt{2}X = Y\sqrt{2}$			**6**
	$\Rightarrow \left(\sqrt{2}X + 1\right)^2 - 1 = Y\sqrt{2}$	**M1A1**	for completing the square; correct **ANSWER GIVEN** obtained	

This line has axis of symmetry $X = \dfrac{-1}{\sqrt{2}}$ **M1** Stated or noted (explicitly) on sketch

$\Rightarrow \dfrac{x+y}{\sqrt{2}} = \dfrac{-1}{\sqrt{2}}$ i.e. $x + y = -1$ **A1**



G1 red curve correct

G1 purple curve and blue line obviously a rotation thro' 45° of their red curve

G1 ALL correct (incl. thro' O)

If red curve does not appear …

G3 for purple curve & blue line correct

G1 for obviously rotated 45° c/w

G0 for anything else

$\boxed{5}$

SI 2013 Mark Scheme Q6

(∗) Coefft. of x^r in $(1+x)^{n+1}$ is $\binom{n+1}{r}$ **B1**

Coefft. of x^r in $(1+x)(1+x)^n$ is from

$$(1+x)\left(\;.....+\binom{n}{r-1}x^{r-1}+\binom{n}{r}x^r+.....\;\right)$$ **M1**

$$\Rightarrow \binom{n+1}{r}=\binom{n}{r-1}+\binom{n}{r}$$ **A1** **GIVEN ANSWER** legitimately obtained

3

For n even, writing $n = 2m$ … **M1** for attempting the even case

$$B_{2m}+B_{2m+1}=\binom{2m}{0}+\binom{2m-1}{1}+\binom{2m-2}{2}+.....+\binom{m+1}{m-1}+\binom{m}{m}$$ **B1**

$$+\binom{2m+1}{0}+\binom{2m}{1}+\binom{2m-1}{2}+\binom{2m-2}{3}+.....+\binom{m+1}{m}$$ **B1**

$$=\binom{2m+1}{0}+\left[\binom{2m}{0}+\binom{2m}{1}\right]+\left[\binom{2m-1}{1}+\binom{2m-1}{2}\right]+.....+\left[\binom{m+1}{m-1}+\binom{m+1}{m}\right]+\binom{m}{m}$$ **M1** for suitable pairings (clear)

$$=\binom{2m+2}{0}+\left[\binom{2m+1}{1}\right]+\left[\binom{2m}{2}\right]+.....+\left[\binom{m+2}{m}\right]+\binom{m+1}{m+1}$$ **M1** for use of first result, **(∗)**

using the result **(∗)** from above and since $\binom{2m+1}{0}=\binom{2m+2}{0}=\binom{m}{m}=\binom{m+1}{m+1}=1$ **M1** for noting the equality of the "1"s

$$=\sum_{j=0}^{m+1}\binom{2(m+1)-j}{j}=B_{2m+2}$$ **A1** Legitimately shown

7

For n odd, writing $n = 2m + 1$ … **M1** for attempting the odd case

$$B_{2m+1} + B_{2m+2} = \binom{2m+1}{0} + \binom{2m}{1} + \binom{2m-1}{2} + \ldots + \binom{m+2}{m-1} + \binom{m+1}{m}$$

Note that this appeared above

$$+ \binom{2m+2}{0} + \binom{2m+1}{1} + \binom{2m}{2} + \binom{2m-1}{3} + \ldots + \binom{m+2}{m} + \binom{m+1}{m+1}$$

B1

$$= \binom{2m+2}{0} + \left[\binom{2m+1}{0} + \binom{2m+1}{1}\right] + \left[\binom{2m}{1} + \binom{2m}{2}\right] + \ldots + \left[\binom{m+2}{m-1} + \binom{m+2}{m}\right] + \left[\binom{m+1}{m} + \binom{m+1}{m+1}\right]$$

M1 for suitable pairings (clear)

$$= \binom{2m+3}{0} + \left[\binom{2m+2}{1}\right] + \left[\binom{2m+1}{2}\right] + \ldots + \left[\binom{m+3}{m}\right] + \binom{m+2}{m+1}$$

M1 for use of first result, **(∗)**

using the result **(∗)** from above and since $\binom{2m+2}{0} = \binom{2m+3}{0} = 1$ **M1** for noting the equality of the "1"s

$$= \sum_{j=0}^{m+1} \binom{2(m+1)+1-j}{j} = B_{2m+3}$$

A1 Legitimately shown

6

B_0 , B_1 = $\binom{0}{0}, \binom{1}{0} = 1, 1$ **B1** Evaluating B_0 , B_1

Evaluating F_0 , F_1 & $F2$ **B1** Both sides must be evaluated, not just stated

Statement that $B_n = F_{n+1}$ **B1** At any point

For a clear justification of result (e.g. inductively) **E1**

(i.e. comment that since $B_0 = F_1$, $B_1 = F_2$ and B_n & F_n satisfy the same recurrence relation, we must have $B_n = F_{n+1}$ for all n)

4

SI 2013 Mark Scheme Q7

(i)	$y = ux \Rightarrow \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$	**B1**	
	$\Rightarrow u + x\dfrac{du}{dx} = \dfrac{1}{u} + u$	**M1**	substituted in
	$\Rightarrow \int u\,du = \int \dfrac{1}{x}\,dx$	**M1**	variables separated and integration attempted
	$\Rightarrow \frac{1}{2}u^2 = \dfrac{y^2}{2x^2} = \ln x \quad (+C)$	**M1**	substituted back for x and y
	$\Rightarrow y^2 = x^2(2\ln x + 2C)$		
	$x = 1,\ y = 2$ substd. to determine C	**M1**	$C = 2$ (the lack of a constant of integration earlier $\Rightarrow$ M0 here)
	$\Rightarrow y = x\sqrt{2\ln x + 4}$		**ANSWER GIVEN** (all working and signs must be correct throughout)
		E1	Justification of $+_{ve}$ square-root required: $y > 0$ when $x = 1$ gives this (Given $x > e^{-2}$, so square-rooting valid – this does not need to be stated by candidates)
			6
(ii)	$y = ux$ used to get $u + x\dfrac{du}{dx} = \dfrac{1}{u} + 2u$	**B1**	
	$\Rightarrow \int \dfrac{2u}{1+u^2}\,du = \int \dfrac{2}{x}\,dx$	**M1**	variables separated and integration attempted
	$\Rightarrow \ln(1+u^2) = 2\ln x \quad (+\ln A)$	**A1**	
	$x = 1,\ y = 2$ substd. to determine A	**M1**	$A = 5$ (the lack of a constant of integration earlier $\Rightarrow$ M0 here)
	$1 + \dfrac{y^2}{x^2} = 5x^2$	**M1**	for eliminating u and correct use of log laws (*with* correct number of terms)
	$y = x\sqrt{5x^2 - 1}$	**A1**	No need to justify choice of $+_{ve}$ square-root, but don't allow $\pm$
	for $x > \frac{1}{\sqrt{5}}$	**B1**	**FT** sensible domains (provided $x > 0$)

(ii) **ALTERNATIVE**

	$y = ux^2$ used to get $2ux + x^2\dfrac{du}{dx} = \dfrac{1}{ux} + 2ux$	**B1M1**	$\dfrac{dy}{dx}$ correct (LHS); full substitution into given d.e.
	$\Rightarrow \int u\,du = \int \dfrac{1}{x^3}\,dx$	**M1**	variables separated and integration attempted
	$\Rightarrow \frac{1}{2}u^2 = \dfrac{-1}{2x^2}\ \left(+\frac{1}{2}C\right)$		
	$x = 1,\ y = 2\ (u = 2)$ substd. to find C	**M1**	$C = 15$ (the lack of a constant of integration earlier $\Rightarrow$ M0 here)
	$\dfrac{y^2}{x^4} = 5 - \dfrac{1}{x^2}$	**M1**	for eliminating u
	$y = x\sqrt{5x^2 - 1}$	**A1**	No need to justify choice of $+_{ve}$ square-root, but don't allow $\pm$
	for $x > \frac{1}{\sqrt{5}}$	**B1**	**FT** sensible domains (provided $x > 0$)
			7

(iii)	$y = ux^2$ used to get $2ux + x^2\dfrac{du}{dx} = \dfrac{1}{u} + 2ux$	**B1M1**	$\dfrac{dy}{dx}$ correct (LHS); full substitution into given d.e.
	$\Rightarrow \int u\,du = \int \dfrac{1}{x^2}\,dx$	**M1**	variables separated and integration attempted
	$\Rightarrow \frac{1}{2}u^2 = \dfrac{-1}{x}\ (+D)$		
	$x = 1,\ y = 2\ (u = 2)$ substd. to find D	**M1**	$D = 3$ (the lack of a constant of integration earlier $\Rightarrow$ M0 here)
	$\dfrac{y^2}{x^4} = 6 - \dfrac{2}{x}$	**M1**	for eliminating u
	$y = x\sqrt{6x^2 - 2x}$ or $y = x\sqrt{2x}\sqrt{3x-1}$	**A1**	No need to justify choice of $+_{ve}$ square-root, but don't allow $\pm$
	for $x > \frac{1}{3}$	**B1**	**FT** sensible domains (provided $x > 0$)
			7

SI 2013 Mark Scheme Q8

(i)

function	domain	range		
$\mathrm{cb}(x) = 2\ln x$	$x > 0$	$\mathbb{R}$	**B1**	
$\mathrm{ab}(x) = (\ln x)^2$	$x > 0$	$y \geq 0$	**B1**	
$\mathrm{da}(x) = \lvert x \rvert$	$\mathbb{R}$	$y \geq 0$	**B1**	Do not allow x or $\sqrt{x^2}$ here; but $\begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$ is ok
$\mathrm{ad}(x) = x$	$x \geq 0$	$y \geq 0$	**B1**	
				These B1s are for the functions.
			B1B1	1st for all domains correct; 2nd for all ranges correct

6

(ii)

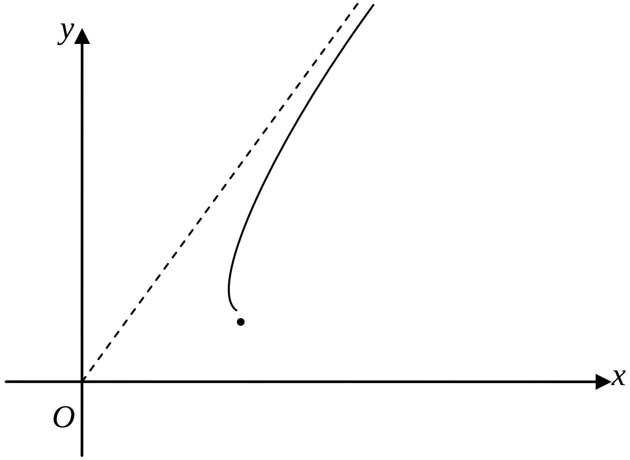
function	domain	range		
$\mathrm{fg}(x) = \lvert x \rvert$ or $\sqrt{x^2}$	$\mathbb{R}$	$y \geq 0$	**B1 B1**	Allow $\begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ but not or $\pm x$
$\mathrm{gf}(x) = \lvert x \rvert$ or $\sqrt{x^2}$	$\lvert x \rvert \geq 1$	$y \geq 1$	**B1 B1**	

In each case, 1st B1 for the fn. & 2nd B1 for both domain and range

OR 1st B1 for both domains correct, 2nd B1 for both ranges correct

4

(iii) $y = \mathrm{h}(x) = x + \sqrt{x^2 - 1}$, $x \geq 1$



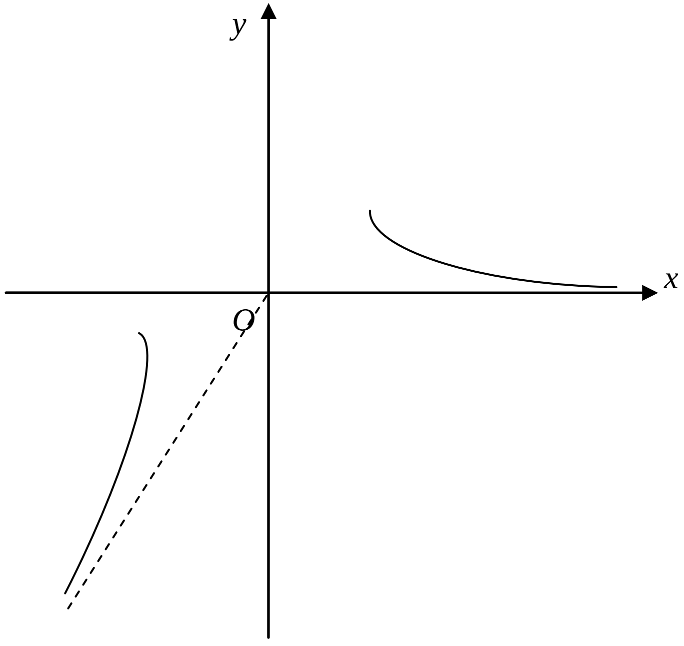
G1 Starts at (1, 1) – noted or clear on sketch

G1 Going upwards $(\rightarrow \infty)$

G1 Approaching asymptote $y = 2x$ (eqn. must be given somewhere)

NB – I will correct the diagram so it is the graph of a *function*

$y = \mathrm{k}(x) = x - \sqrt{x^2 - 1}\,, \;\; |\,x\,| \geq 1$



G1 Two branches

G1 (1, 1) and (–1, –1) noted or clear on sketch

G1 Branch in Q1 approaches (from above) asymptote $y = 0$ (or "x-axis") must be noted somewhere

G1 Branch in Q3 approaches asymptote (from above) $y = 2x$ must be noted somewhere

7

Domain of kh is $x \geq 1$ **B1**

Range of h is $y \geq 1 \Rightarrow$ Range of kh is $0 < y \leq 1$ **M1A1**

3

SI 2013 Mark Scheme Q9

For *A* $\dot{y} = u\sin\alpha - gt$		
For *B* $\dot{y} = v\sin\beta - gt$	**B1**	for both (possibly implicitly)
When $\dot{y} = 0$, $t = \dfrac{u\sin\alpha}{g} = \dfrac{v\sin\beta}{g}$	**M1**	**OR** Noting that vertical speeds must be equal for collision; i.e. $u\sin\alpha - gt = v\sin\beta - gt$
giving $u\sin\alpha = v\sin\beta$	**A1**	
Noting that speeds *after* collision must equal those before (for each particle)	**B1**	
<u>**CLM**</u> $mu\cos\alpha - Mv\cos\beta = M\,w_B - m\,w_A$	**M1 A1**	Not required to have w_B and w_A in final correct form here
$\Rightarrow mu\cos\alpha = Mv\cos\beta$	**A1**	
Dividing the two results	**M1**	
$m\cot\alpha = M\cot\beta$	**A1**	**ANSWER GIVEN**

9

Use of times when $\dot{y} = 0$, $t = \dfrac{u\sin\alpha}{g} = \dfrac{v\sin\beta}{g}$ in displacement expressions	**M1**	
$x_A = b = \dfrac{u^2\sin\alpha\cos\alpha}{g}$ and $x_B = (d -)\dfrac{v^2\sin\beta\cos\beta}{g}$	**A1**	both
So $d = b + \dfrac{1}{g}(v\sin\beta)(v\cos\beta)$	**M1**	
$= b + \dfrac{1}{g}(u\sin\alpha)\left(\dfrac{m}{M}u\cos\alpha\right)$	**M1 M1**	for use of previously obtained results
$= b + \dfrac{m}{M}b$ or $\dfrac{b(m+M)}{M}$	**A1**	
giving $b = \dfrac{Md}{m+M}$	**A1**	**ANSWER GIVEN**

7

Ht. at collision is given by $y = \dfrac{u^2 \sin^2 \alpha}{2g}$ **M1 A1**

i.e. $h = \dfrac{1}{2} \times \dfrac{u^2 \sin\alpha \cos\alpha}{g} \times \dfrac{\sin\alpha}{\cos\alpha}$ **M1**

$= \tfrac{1}{2} b \tan\alpha$ **A1**

4

SI 2013 Mark Scheme Q10

Resolving ↑ $R = mg$	**B1**	
Friction Law $F = \mu R$ (in motion)	**B1**	
N2L $\mu mg = -ma \Rightarrow a = -\mu g$	**B1**	
By CAF $w_{i+1}{}^2 = v_i{}^2 - 2\mu gd$	**B1**	Can be done by work-energy also
(v_i is speed of puck leaving i-th barrier, $i = 0, 1, 2, \ldots$ and w_i is speed of puck arriving at i-th barrier, $i = 1, 2, 3, \ldots$)		
NEL / NLR $v_{i+1} = r.w_{i+1}$	**M1**	
$\Rightarrow v_{i+1}{}^2 = r^2 v_i{}^2 - 2r^2 \mu gd$	**A1**	**ANSWER GIVEN** any correct form

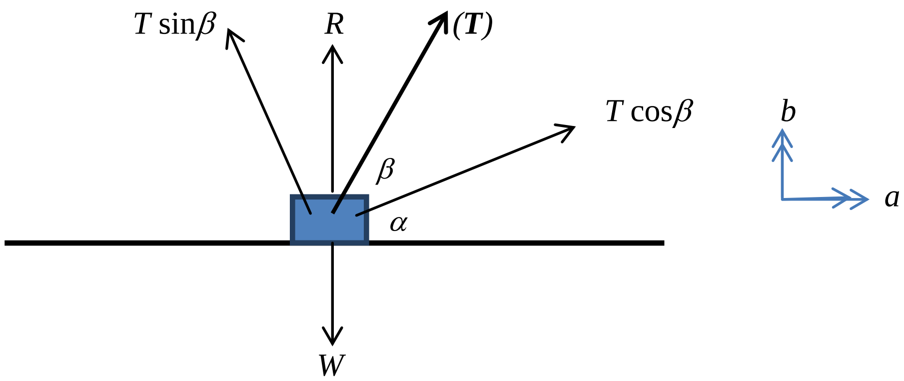
6

$v_1{}^2 = r^2 v^2 - 2r^2 \mu gd$		
$v_2{}^2 = r^4 v^2 - 2r^4 \mu gd - 2r^2 \mu gd$		
$v_3{}^2 = r^6 v^2 - 2r^6 \mu gd - 2r^4 \mu gd - 2r^2 \mu gd$		
$= r^6 v^2 - 2r^2 \mu gd \{1 + r^2 + r^4\}$	**M1**	iterating given result
… … … … … …		
$v_n{}^2 = r^{2n} v^2 - 2r^2 \mu gd \{1 + r^2 + r^4 + \ldots + r^{2n-2}\}$	**A1 A1**	general term; 1st A for $r^{2n} v^2$ and 2nd A for $\{1 + r^2 + r^4 + \ldots + r^{2n-2}\}$ correct (in this form)
$\{1 + r^2 + r^4 + \ldots + r^{2n-2}\} = \dfrac{1 - r^{2n}}{1 - r^2}$ at any stage	**B1**	
Setting $v_n = 0$ (realistic expression)	**M1**	$\dfrac{v^2}{2\mu gd} r^{2n} = \dfrac{(1 - r^{2n}) r^2}{1 - r^2}$
Identifying $r^{2n} \left(= \dfrac{r^2}{r^2 + k(1 - r^2)} \right)$	**M1**	
Taking logs and solving for $n = \dfrac{\ln\left(\dfrac{r^2}{r^2 + k(1 - r^2)}\right)}{2 \ln r}$	**M1 A1**	allow any sensible correct alternative form

8

Setting $r = e^{-1}$ in their result	**M1**		
$\Rightarrow n = -\frac{1}{2}\ln\left(\frac{1/e^2}{1/e^2 + k(1-1/e^2)}\right)$			
$= -\frac{1}{2}\ln\left(\frac{1}{1+k(e^2-1)}\right)$			
$= \frac{1}{2}\ln\left(1+k(e^2-1)\right)$	**A1**	**ANSWER GIVEN** MUST be from correct log. work	**2**
When $r = 1$, $v_{i+1}^{\,2} - v_i^{\,2} = -2r^2\mu gd$	**B1**		
$v_1^{\,2} - v^2 = -2r^2\mu gd$			
$v_2^{\,2} - v_1^{\,2} = -2r^2\mu gd$			
$v_3^{\,2} - v_2^{\,2} = -2r^2\mu gd$			
$\ldots \quad \ldots \quad \ldots$			
$v_n^{\,2} - v_{n-1}^{\,2} = -2r^2\mu gd$	**M1**	iterating given result	
$\Rightarrow v_n^{\,2} - v^2 = -n.2r^2\mu gd$	**A1**		
Setting $v_n^2 = 0 \Rightarrow n = \frac{v^2}{2\mu gd} = k$	**A1**		
ALT: When $r = 1$, distance travelled is just nd	**B2**		
Then $v^2 = 2\mu gnd$	**M1**	(by the CAF)	
and $n = \frac{v^2}{2\mu gd} = k$	**A1**		**4**

SI 2013 Mark Scheme Q11



Since $\alpha + \beta < \frac{1}{2}\pi$, the two tensions $T\sin\beta$ and $T\cos\beta$ are effectively components of a notional force ($\boldsymbol{T}$), inclined slightly to the right of the normal contact reaction force R (as shown). Hence, if there is motion, it will take place to the right (i.e. $a \geq 0$). This is unlikely to be stated explicitly, but must be recognised (esp. in the Resolving/N2L $\rightarrow$ statement).

Also, the acceleration components a and b are not required in (i), but may appear.

<u>Resolving ↑</u> $R + T\sin\beta\cos\alpha + T\cos\beta\sin\alpha = W$	**M1 A1**	
<u>Resolving →</u> $F + T\sin\beta\sin\alpha = T\cos\beta\cos\alpha$	**M1 A1**	(In both cases, the M is earned only if there are the right no. of forces)
$W = R + T\sin(\alpha+\beta)$ and $F = T\cos(\alpha+\beta)$	**M1**	use of trig. identities (at least once)
$W > T\sin(\alpha+\beta) \Rightarrow R > 0$ so block does not rise	**B1**	noted explicitly
For eqlm., $F \leq \mu R$ and using $\mu = \tan\lambda$		
$\Rightarrow T\cos(\alpha+\beta) \leq \tan\lambda\,\{W - T\sin(\alpha+\beta)\}$	**M1**	
i.e. $W\tan\lambda \geq T\tan\lambda\sin(\alpha+\beta) + T\cos(\alpha+\beta)$	**A1**	
$\Rightarrow W\sin\lambda \geq T\sin\lambda\sin(\alpha+\beta) + T\cos(\alpha+\beta)\cos\lambda$ $= T\cos(\alpha+\beta-\lambda)$	**A1**	**ANSWER GIVEN**

$\boxed{\mathbf{9}}$

$W = T\tan\frac{1}{2}(\alpha+\beta) \;\Rightarrow\; R = T\sin\frac{1}{2}(\alpha+\beta) - T\sin(\alpha+\beta) < 0$

so $R = 0$ (and $F = 0$) as block lifts from ground — **E1** explicitly and clearly explained

Taking unit vectors **I** and **j** in directions $\rightarrow$ and $\uparrow$ respectively:

$$\boldsymbol{T_A} = \begin{pmatrix} -T\sin\beta\sin\alpha \\ T\sin\beta\cos\alpha \end{pmatrix},\; \boldsymbol{T_B} = \begin{pmatrix} -T\cos\beta\cos\alpha \\ T\cos\beta\sin\alpha \end{pmatrix},\; \boldsymbol{W} = \begin{pmatrix} 0 \\ -T\tan\frac{1}{2}(\alpha+\beta) \end{pmatrix}$$ **B1 B1 B1**

and the resultant force on the block is $\boldsymbol{T_A} + \boldsymbol{T_B} + \boldsymbol{W}$ — **M1**

$$= \begin{pmatrix} T\cos(\alpha+\beta) \\ T\sin(\alpha+\beta) - T\tan\frac{1}{2}(\alpha+\beta) \end{pmatrix}$$ **A1**

which is in the direction $\tan^{-1}\left(\dfrac{\sin(\alpha+\beta) - \tan\frac{1}{2}(\alpha+\beta)}{\cos(\alpha+\beta)}\right)$ — **M1** (relative to **i**) **NB** $\alpha+\beta = 2\phi$ throughout

$$= \tan^{-1}\left(\frac{2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta) - \dfrac{\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha+\beta)}}{\cos(\alpha+\beta)}\right)$$ **M1** full use of half-angles (in numerator)

$$= \tan^{-1}\left(\frac{\sin\frac{1}{2}(\alpha+\beta)\left\{2\cos^2\frac{1}{2}(\alpha+\beta) - 1\right\}}{\cos(\alpha+\beta)\,.\cos\frac{1}{2}(\alpha+\beta)}\right)$$ **M1**

$$= \tan^{-1}\left(\tan\tfrac{1}{2}(\alpha+\beta)\right)$$ **M1** must come from noting that

$$2\cos^2\tfrac{1}{2}(\alpha+\beta) - 1 = \cos(\alpha+\beta)$$

$$= \tfrac{1}{2}(\alpha+\beta) = \phi$$ **A1** **ANSWER GIVEN**

11

SI 2013 Mark Scheme Q12

(i) $P_3(3) = \frac{9}{9}\times\frac{6}{8}\times\frac{3}{7} = \frac{9}{28}$ **M3 A1 ANSWER GIVEN**

OR if we consider them as 3R, 3B, 3G tablets:

$\frac{3}{9}\times\frac{3}{8}\times\frac{3}{7} = \frac{3}{56}$ for one specified order (e.g. RBG) **M1 A1**

$\times$ 3! = 6 for no. of perms of the 3 colours = $\frac{9}{28}$ **M1 A1 ANSWER GIVEN**

OR No. of correct selections = $\frac{9}{1}\times\frac{6}{2}\times\frac{3}{3} = 27$ **B1**

Total no. of selections = $\frac{9}{1}\times\frac{8}{2}\times\frac{7}{3} = 84$ **B1**

Dividing $\Rightarrow$ $\frac{27}{84} = \frac{9}{28}$ **M1 A1 ANSWER GIVEN**

As all days equivalent, prob. for 3rd day is also $\frac{9}{28}$ **B1** however gained

5

(ii) Using the 1st method above:

$$P_3(n) = \frac{3n}{3n}\times\frac{2n}{3n-1}\times\frac{n}{3n-2} = \frac{2n^2}{(3n-1)(3n-2)}$$ **M1 M1 M1 A2**

Using the 2nd method above:

$$P_3(n) = \frac{n}{3n}\times\frac{n}{3n-1}\times\frac{n}{3n-2}\times 3! = \frac{2n^2}{(3n-1)(3n-2)}$$ **M1 M1 M1 M1 A1** or $\frac{n^3}{\binom{3n}{3}}$

5

(iii) P(correct tablet on each of the n days)

$= P_2(n) \times P_2(n-1) \times P_2(n-2) \times \ldots P_2(2) \times P_2(1)$ **M1**

$$= \left[\frac{n^2 . 2!}{2n(2n-1)}\right] \times \left[\frac{(n-1)^2 . 2!}{(2n-2)(2n-3)}\right] \times \left[\frac{(n-2)^2 . 2!}{(2n-4)(2n-5)}\right] \times \ldots \times \left[\frac{2^2 . 2!}{4.3}\right] \times \left[\frac{1^2 . 2!}{2.1}\right]$$ **A1 A1 A1 … A1**

$$= \frac{(n!)^2 2^n}{(2n)!} \text{ or } \frac{2^n}{\binom{2n}{n}}$$ **A1** factorials must appear at some stage

---

OR Prob. $= \left(1 \times \frac{n}{2n-1}\right) \times \left(1 \times \frac{n-1}{2n-3}\right) \times \left(1 \times \frac{n-2}{2n-5}\right) \times \ldots \times \left(1 \times \frac{2}{3}\right) \times \left(1 \times \frac{1}{1}\right)$ **M1 A1 A1 A1 … A1**

$$= \frac{n!}{(2n-1)(2n-3)\ldots 5.3.1} = \frac{(n!)^2 2^n}{(2n)!}$$ **A1** factorials must appear at some stage

---

Using $n! \approx \sqrt{2n\pi}\left(\frac{n}{e}\right)^n$ and $(2n)! \approx \sqrt{4n\pi}\left(\frac{2n}{e}\right)^{2n}$ **M1 M1**

$$\Rightarrow \text{Prob.} = \frac{2n\pi \times \frac{n^{2n}}{e^{2n}} \times 2^n}{\sqrt{4n\pi} \times \frac{2^{2n} n^{2n}}{e^{2n}}}$$ **A1** correct powers of n, 2 and e

$$= \frac{2n\pi \times 2^n}{2\sqrt{n\pi} \times 2^{2n}} = \frac{\sqrt{n\pi}}{2^n}$$ **A1** **ANSWER GIVEN**

10

SI 2013 Mark Scheme Q13

For $x \in \{0, 1, 2, 3\}$ **B1** **1**

For each of the first three cases (the easier ones are $X = 0$ and $X = 3$) that they evaluate, I am offering 5 marks.
The fourth case scores only 1 mark, **FT**, as the result of 1 – (the sum of their 3 other cases) … provided the sum < 1, of course.

P(X = 0): the 7 pairs from which a singleton is chosen in ${}^{26}C_7$ ways **M1**
choosing one from each pair in 2^7 ways **M1**

$\Rightarrow$ P(X = 0) = $\dfrac{{}^{26}C_7 \times 2^7}{{}^{52}C_7}$ **M1**

Use of ${}^{26}C_7 = \dfrac{26.25.24.23.22.21.20}{7.6.5.4.3.2.1}$ & ${}^{52}C_7 = \dfrac{52.51.50.49.48.47.46}{7.6.5.4.3.2.1}$ **M1***

P(X = 0) = $\dfrac{3520}{5593}$ **A1**

NB this is a floating M mark that can be awarded, at any stage of a largely unsuccessful solution, for both ${}^{52}C_7$ and some ${}^{26}C_r$ (r = 3, 4, 5, 6, 7) correct. **5**

P(X = 3): the 3 pairs from 26 can be chosen in ${}^{26}C_3$ ways **M1**
the one singleton from the remaining 23 pairs in ${}^{23}C_1 = 23$ ways **M1**
and the one from that pair in 2^1 ways **M1**

$\Rightarrow$ P(X = 3) = $\dfrac{{}^{26}C_3 \times {}^{23}C_1 \times 2^1}{{}^{52}C_7}$ **M1**

P(X = 3) = $\dfrac{5}{5593}$ **A1** **5**

$P(X = 1)$: the 1 pair can be chosen in ${}^{26}C_1 = 26$ ways **M1**

the 5 pairs from which a singleton is chosen in ${}^{25}C_5$ ways **M1**

and the singletons from those pairs in 2^5 ways **M1**

$\Rightarrow P(X = 1) = \dfrac{26 \times {}^{25}C_5 \times 2^5}{{}^{52}C_7}$ **M1**

$P(X = 1) = \dfrac{1848}{5593}$ **A1** or $\dfrac{264}{799}$

5

$P(X = 2)$: the 2 pairs can be chosen in ${}^{26}C_2$ ways **M1**

the 3 pairs from which a singleton is chosen in ${}^{24}C_3$ ways **M1**

and the singletons from those pairs in 2^3 ways **M1**

$\Rightarrow P(X = 2) = \dfrac{{}^{26}C_2 \times {}^{24}C_3 \times 2^3}{{}^{52}C_7}$ **M1**

$P(X = 2) = \dfrac{220}{5593}$ **A1**

(NB – 4th case scores just … **B1**)

1

$E(X) = \sum x\,.\,p(x) = \dfrac{1}{5593}(0 \times 3520 + 1 \times 1848 + 2 \times 220 + 3 \times 5)$ **M1 A1** correct unsimplified

$= \dfrac{2303}{5593} = \dfrac{7 \times 7 \times 47}{7 \times 17 \times 47} = \dfrac{7}{17}$ **A1** **ANSWER GIVEN** (working must be shown)

3

NB $E(X)$ = Prob. of given pair occurring $\times$ No. of possible pairs = $\left(\dfrac{7}{52} \times \dfrac{6}{51}\right) \times 26 = \dfrac{7}{17}$